A Mathematical Model for Indian Ocean Circulation in Spherical Coordinate

Ghazi Mirsaeid, Mojgan1; Mohammad, Mehdizadeh Mehdi1*; Bannazadeh, Mohammad Reza2

1- Department of Physical Oceanography, Faculty of Marine Science and Technology, Hormozgan University, Hormozgan, IR Iran
2- Erfan University

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Abstract
In recent years, the Indian Ocean (IO) has been discovered to have a much larger impact on climate variability than previously thought. This paper reviews processes in which the IO is, or appears to be, actively involved. We begin the mathematical model with a pattern for summer monsoon winds. Three dimensional temperature and velocity fields are calculated analytically for the ocean forced by wind stress and surface heat flux. A basic thermal state involving a balance of lateral and vertical heat diffusion is assumed. The wind stress is chosen such that a tropical mass transport gyre is generated. An effect of nonlinear heat advection is calculated by a perturbation method. The zero order temperature field gives a rough overall representation of oceanic thermocline. A baroclinic eastward flow in the upper part, with a westward return flow below is associated with this field. This circulation is closed through thin up and downwelling layers at the sides. Superimposed, there is a barotropic wind driven circulation, with a transport field of the type described by Munk. The interior temperature field to the next order is affected, not only by interior heat advection but also by heat advection in the Ekman layer, in the up and downwelling layers and in the main western boundary current.

Keywords: Spherical coordinate, Heat advection, Circulation, Indian Ocean

1. Introduction
Several review papers have been published previously on IO circulation. Schott and McCreary [2001] reviewed the current state of knowledge on the monsoon circulation, both with regard to recent observations and to the hypotheses put forward for their interpretation. They summarize, and try to reconcile, classical concepts and interpretations with the newly available observations. Their focus is on the monsoon circulation north of about 10 S. Schott, et al. (2009) reviewed climate phenomena and processes in which the IO is or appears to be actively involved.
We begin with an update of the IO mean circulation and monsoon system. It is followed by reviews of ocean/atmosphere phenomenon at intra-seasonal, inter-annual, and longer time scales. Much
of above reviews address the two important types of inter-annual variability in the IO, El Niño–Southern Oscillation (ENSO) and the recently identified Indian Ocean Dipole (IOD).

Muni et al., (2014) studied Tropical Indian Ocean Surface and Subsurface Temperature Fluctuations in a Climate Change Scenario. The feature and evolution mechanisms of the tropical Indian Ocean temperature between pre-warming and warming periods and also during IOD events that co-occurred with El Niño are studied using Simple Ocean Data Assimilation (SODA) data set. During the positive IOD with co-occurred El Nino years in a climate change period (1970-2008) surface warm temperatures are extended from the Sumatra region to off the African coast in pre-monsoon season. But, the subsurface temperature shows different pattern, the warming is more in the central Arabian Sea and south Bay of Bengal area.

Our review is covering the description and understanding of physical oceanography concepts by mathematical method. In later experiments, the temptation is to improve oceanic investigation by numerical models by bringing in new effects, such as bottom topography, nonlinear equation of state, etc. The repetitions of experiments over a range of parameter values which give a better insight into the nature of simpler problem are often neglected. Thus, there seems to be some gaps in our basic theoretical knowledge which needs to be filled, by a combination of analytical theory and relatively simple numerical models.

The problems studied here involve the calculation of the three dimensional temperature and velocity field in a bounded, rectangular ocean on the real earth, forced by a meridionally varying wind stress and surface heat flux. The motion and driving forces are taken to be independent of time. In this paper, the equations of motion are approximated and simplified. After averaging the equations of motion, the Reynold stresses are defined. Phenomenological transfer coefficients associated with the horizontal and vertical eddy motion are assumed to be constants, with the same values for the eddy viscosity and the eddy diffusion assumed. The eddy coefficients in the real ocean most likely vary with position; but since the main purpose of this paper is the investigation of the general behavior of a model ocean, the use of constant eddy coefficients is considered to be sufficient for the present. We note that, at present, no theory exists which predicts the variable eddy coefficients in the ocean. Salinity is represented through an equivalent temperature effect. At the solid boundaries, the normal, baroclinic, vertically averaged flow must vanish; hence, boundary layers must form. Actually, this circulation is closed principally through thin upwelling and downwelling layers next to the walls. The barotropic part is found from the mass transport equation. Since, bottom stresses have a negligible effect relative to that of lateral friction in our case, the linear mass transport equation is of the type given by Munk. The solution to this equation can be found analytically and analytic theory can also be given through a perturbation calculation. The zero order temperature field gives a rough overall representation of oceanic thermocline. The first order calculation of temperature includes, not only the advective effects of the interior flow, but equally important effects of thermal advection in the Ekman surface layer and in the barotropic and baroclinic side layers.

2. Data and method

2.1. Basic Equations

Consider a rectangular oceanic basin in spherical coordinate (Fig. 1). At the upper boundary a zonal wind stress and a normal heat flux (counted positive when directed upward) are applied, i.e.

\[
\tau^\lambda = -\tau_0 \sin(k\pi \phi)
\]

\[
Q = -Q_0 \cos\frac{\pi}{2} \phi
\]

At the meridional walls, \(\lambda=0, 1\), it is required that the horizontal velocity and the normal heat flux vanish, while the total velocity and the vertical heat
flux vanish at the bottom. At the top boundary, the vertical velocity vanishes, while the momentum flux and heat flux are specified according to the boundary conditions (1), (2).

\( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{\partial}{\partial z} (\mathbf{V} \cdot \mathbf{w}) = \frac{1}{R} \frac{\partial p}{\partial \lambda} \)

Fig. 1: Geometry of studied basin.

The density variations are assumed small enough to allow the Boussinesq approximation to be valid. It is assumed that the turbulent friction and heat diffusion can be described by use of constant eddy coefficients, and that the turbulence is horizontally isotropic, allowing the use of only one coefficient in the horizontal plane. Fluxes due to molecular diffusion are considered negligible relative to the turbulent eddy fluxes.

In accordance with the above assumptions of the problem, the following equations can be written, expressing the horizontal momentum balance, hydrostatic balance, incompressibility, and heat flux divergence balance, respectively:

\( \lambda \) direction: \( \rho \left( \frac{D}{Dt} \frac{\partial u}{\partial t} + \frac{u \partial u}{\partial x} - 2 \Omega \sin \varphi u + 2 \Omega \cos \varphi w \right) = -\frac{1}{R \cos \varphi} \frac{\partial p}{\partial \lambda} \) \hspace{1cm} (3)

\( \phi \) direction: \( \rho \left( \frac{D}{Dt} \frac{\partial v}{\partial t} + \frac{u \partial v}{\partial x} + 2 \Omega \sin \varphi u \right) = \frac{1}{R} \frac{\partial p}{\partial \varphi} \) \hspace{1cm} (4)

r direction: \( \frac{\partial p}{\partial z} \equiv g \alpha T \)

\( \frac{1}{R^2} \frac{\partial}{\partial z} (R^2 w) + \frac{1}{R \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{1}{R \cos \varphi} \frac{\partial u}{\partial \lambda} = 0 \) \hspace{1cm} (5)

Where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{\partial}{\partial z} (\mathbf{V} \cdot \mathbf{w}) = \frac{1}{R} \frac{\partial p}{\partial \lambda} \)

The dependent variables are prescribed such that \( u, v, \) and \( w \) are the velocity components in the \( \lambda, \varphi, \) and \( r \) directions, \( P \) states the pressure, \( \rho \) denotes the density, \( T \) is the temperature, and \( g \) denotes the acceleration of gravity.

After using derivation of stress tensor and Prandtl’s Mixing-Length theory and imposing rules of operating on mean time-averaged and some simplifications, the results for momentum balance are:

\( \rho \left( \frac{D}{Dt} \frac{\partial u}{\partial t} - \frac{u \partial u}{\partial x} - 2 \Omega \sin \varphi u - 2 \Omega \cos \varphi w \right) = -\frac{1}{R \cos \varphi} \frac{\partial p}{\partial \lambda} \)

\( A \left( \nabla^2 U \right) = \frac{1}{R} \frac{\partial p}{\partial \varphi} \)

\( \rho \left( \frac{Dv}{Dt} + \frac{u^2 \partial v}{\partial x} + 2 \Omega \sin \varphi u \right) = \frac{1}{R} \frac{\partial p}{\partial \varphi} \)

\( \lambda \) direction: \( \frac{D}{Dt} \left( \frac{\partial u}{\partial t} \right) + \frac{1}{R} \left( \frac{\partial u}{\partial \varphi} \right) + \frac{1}{R} \left( \frac{\partial u}{\partial \lambda} \right) = 0 \) \hspace{1cm} (6)

And heat balance:

\( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla T + A \frac{\partial^2}{\partial z^2} \)

\( \nabla^2 = \frac{1}{R \cos \varphi} \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial}{\partial \varphi} \right) + \frac{1}{R^2 \sin^2 \varphi} \frac{\partial}{\partial \lambda} \) \hspace{1cm} (13)

Boundary conditions become:

\( A \frac{\partial u}{\partial z} = \tau_z (\varphi), A \frac{\partial T}{\partial z} = -Q(\varphi), \) at \( z = 0 \) \hspace{1cm} (14-1)

\( \frac{\partial v}{\partial z} = w = 0 \) at \( z = 0 \) \hspace{1cm} (14-2)

\( U = V = w = \frac{\partial T}{\partial z} = 0 \) at \( z = -H \) \hspace{1cm} (14-3)

\( U = V = \frac{1}{R \cos \varphi} \frac{\partial T}{\partial \lambda} = 0 \) at \( \lambda = 0.1 \) \hspace{1cm} (14-4)

\( \frac{\partial u}{\partial \varphi} = V = \frac{1}{R \cos \varphi} \frac{\partial T}{\partial \varphi} = 0 \) at \( \varphi = 0.1 \) \hspace{1cm} (14-5)

2.2. Non-dimensionalization and expansion:

The following transformation to non-dimension variables is introduced:
The scale depth $D$ (thermocline depth) is set by balancing the horizontal and vertical heat diffusion, assuming the same typical temperature variation, in both directions. The thermal boundary condition and the thermal wind relation then determines the amplitudes $\Delta T$ and $U_1$. The relations take the form:

$$A_R \frac{\Delta T}{D^2} = A_V \frac{\Delta V}{D^2} \Rightarrow D = \frac{A_R}{\sqrt{A_V}} \frac{R}{D} \quad (16)$$

$$Q_0 = A_V \frac{\Delta V}{D} \Rightarrow \Delta T = \frac{DQ_0}{A_V} = \frac{RQ_0}{(A_H A_V)^{1/2}} \quad (17)$$

$$\frac{f_0U_t}{D} = \frac{gR \Delta T}{R} \Rightarrow U_1 = \frac{gR \Delta T}{A_H f_0} \quad (18)$$

The non-dimensionalized equations take the corresponding form:

$$R_0 \left( \frac{U}{\cos \phi} \frac{\partial U}{\partial \lambda} + \frac{V}{\cos \phi} \frac{\partial V}{\partial \phi} + \frac{W}{\cos \phi} \frac{\partial W}{\partial Z} - \frac{U}{\cos \phi} V \tan \phi \right) - V \sin \phi$$

$$+ \frac{D}{R} \frac{\cos \phi}{\cos \phi} = E \nabla^2 U - E \left( \frac{\partial U}{\partial \phi} + \frac{1}{\cos \phi} \frac{\partial V}{\partial \phi} \right) \tan \phi$$

$$- \frac{1}{\cos \phi} \frac{\partial P}{\partial \phi} \tan \phi$$

$$- \frac{1}{\cos \phi} \frac{\partial P}{\partial \phi} \tan \phi$$

$$R_0 \left( \frac{U}{\cos \phi} \frac{\partial V}{\partial \phi} + \frac{V}{\cos \phi} \frac{\partial V}{\partial \phi} + \frac{W}{\cos \phi} \frac{\partial W}{\partial Z} + U^2 \tan \phi \right) + U \sin \phi = E (\nabla^2 V) + \left( \frac{2}{\cos \phi} \frac{\partial V}{\partial \phi} \right) \tan \phi - \frac{\partial P}{\partial \phi} \tan \phi$$

$$\frac{\partial P}{\partial Z} \cong T$$

$$- V_0 \sin \phi + \frac{D}{R} - \cos \phi = E (\nabla^2 U_0) - \left( \frac{\partial V_0}{\partial \phi} + \frac{1}{\cos \phi} \frac{\partial V_0}{\partial \phi} \right) \tan \phi - \frac{1}{\cos \phi} \frac{\partial P_0}{\partial \phi} \tan \phi$$

$$U_0 \sin \phi = E (\nabla^2 V_0) + \left( \frac{2}{\cos \phi} \frac{\partial U_0}{\partial \phi} \right) \tan \phi - \frac{\partial P_0}{\partial \phi} \tan \phi$$

$$\nabla^2 T_0 \cong 0$$

Where the nondimensional numbers appearing in these equations are:

$$R_0 = \frac{U_t}{f_0 R} = \frac{g \alpha}{A_H f_0^2}$$

$$E = \frac{A_H}{f_0 R^2} = \frac{A_V}{f_0 D^2}$$

$$\sigma = \frac{\tau_0 f_0}{\frac{1}{\sqrt{A_H}} \frac{A_H}{A_V}} \left[ \text{ratio of wind forcing to thermal forcing} \right] \quad (26)$$

$$\delta = \frac{D}{H} \left( \frac{A_V}{A_H} \right)^{1/2} \frac{R}{H} \left[ \text{relative thermocline depth} \right] \quad (27)$$

The Rossby and Ekman numbers defined above are very small for the real oceans, and it seems natural to try to find a solution in terms of a series developed from these. Using $R_0 E^{-1}$ as the expansion parameter, we can write like the following series approximations:

$$T = T_0 + R_0 E^{-1} T_1 + (R_0 E^{-1})^2 T_2 + \cdots \quad (28)$$

This makes the zero-order thermal balance diffusive. The nonlinear heat advection appears in the first order correction, while the dynamics remain linear.

We can rewrite $\sigma$ as bellow:

$$\sigma = \frac{\tau_0 f_0}{\frac{1}{\sqrt{A_H}} \frac{A_H}{A_V}} \left[ \text{ratio of wind forcing to thermal forcing} \right]$$

$$\sigma^* = \frac{\tau_0 f_0}{\frac{1}{\sqrt{A_H}} \frac{A_H}{A_V}} \left[ \text{ratio of wind forcing to thermal forcing} \right]$$

The number $\sigma^*$ gives the ratio of the Ekman wind drift to the thermal wind transport over the thermocline depth. Under the above assumption the zero-order problem is described by:

$$- V_0 \sin \phi + \frac{D}{R} - \cos \phi = E (\nabla^2 U_0) -$$

$$\left( \frac{\partial V_0}{\partial \phi} + \frac{1}{\cos \phi} \frac{\partial V_0}{\partial \phi} \right) \tan \phi - \frac{1}{\cos \phi} \frac{\partial P_0}{\partial \phi} \tan \phi$$

$$U_0 \sin \phi = E (\nabla^2 V_0) + \left( \frac{2}{\cos \phi} \frac{\partial U_0}{\partial \phi} \right) \tan \phi - \frac{\partial P_0}{\partial \phi} \tan \phi$$

$$\nabla^2 T_0 \cong 0$$

With the nonhomogeneous boundary conditions:

$$E \frac{\partial U_0}{\partial Z} = \sigma^* \phi \sin \pi \phi$$

$$\frac{\partial T_0}{\partial Z} = \cos \frac{\pi \phi}{2}$$

Both prescribed at $z=0$, and with homogeneous boundary conditions:

$$U_0 = V_0 = W_0 = \frac{\partial T_0}{\partial Z} = 0 \quad \text{at} \quad z = -\frac{H}{D} = -\frac{1}{\delta} \quad (38)$$

$$U_0 = V_0 = \frac{1}{R \cos \phi} \frac{\partial T_0}{\partial \phi} = 0 \quad \text{at} \quad \lambda = 0, 1 \quad (39)$$
3. Results

3.1. The Zero Order Solution

The solution to the heat diffusion equation with the top boundary conditions is independent of \( \lambda \). We can write:

\[
\frac{1}{R} \frac{\partial U_0}{\partial \phi} = V_0 = \frac{1}{R} \frac{\partial T_0}{\partial \phi} = 0 \quad \text{at } \phi = 0,1 \tag{40}
\]

\[
W_0 = \frac{\partial V_0}{\partial Z} = 0 \quad \text{at } z = 0 \tag{41}
\]

The first term, horizontally the solution varies as \( \cos \frac{\pi \phi}{Z} \); and the vertical variation is of the form \( Ae^{\pm \pi Z/2} + Be^{-\pi Z/2} \). Second term is a summation which is due to the assumed spheroid of the earth’s surface. \( c_0(n,m) \) is obtained by substituting (46) into (42), Fourier decomposing the equation in the \( \phi \) and \( Z \) directions and solving the resultant relations for \( c_0(n,m) \). The \( T_0 \) solution is depicted in Figure 2.

3.2. Velocity Field

The discussion of the zero order velocity field starts with the surface regime.

\[
\tau_\lambda = E \frac{\partial U_0}{\partial Z} = -\sigma^* \phi \sin \kappa \phi \tag{47}
\]

Since the forcing function is independent of \( \lambda \), the velocity in the Ekman layer is independent of \( \lambda \). By equation (33), we have \( \frac{\partial P_0}{\partial Z} = T_0 \), since \( T_0 \) is independent of \( \lambda \) so \( \frac{\partial P_0}{\partial \lambda} = 0 \). Then, the equation of motion can be written as follows:

\[
-v_0 \sin \phi = E \frac{\partial^2 U_0}{\partial Z^2} \tag{48}
\]

\[
-u_0 \sin \phi = E \frac{\partial^2 V_0}{\partial Z^2} - \frac{\partial P_0}{\partial \phi} \tag{49}
\]

\[
\frac{\partial P_0}{\partial Z} \equiv T_0 \tag{50}
\]

\[
\frac{\partial W_0}{\partial Z} + \frac{1}{\cos \phi} \frac{\partial (V_0 \cos \phi)}{\partial \phi} = 0 \tag{51}
\]
Integrating the above equations and using the stress condition yields:

\[-v_E \sin \phi = \tau_\lambda \]

(52)

\[W_E = \frac{1}{\cos \phi} \frac{\partial (v_E \cos \phi)}{\partial \phi} \]

(53)

Where, \(v_E\) is the meridional Ekman transport and \(W_E\) is the velocity at the bottom Ekman layer. By combining equations (52) and (53), we will have:

\[W_E = -\frac{1}{\cos \phi} \frac{\partial \tau_\lambda \cos \phi}{\partial \phi} \]

(54)

or

\[W_E = \sigma^* \frac{\partial (\sin \phi \sin \phi \cos \phi)}{\partial \phi} \]

(55)

Near the ocean bottom, a second Ekman layer exists. Here, the velocities are of order 1 at most. A second velocity field which extends from \(z=0\) to \(-1\) is superimposed from the Ekman layer and splits up into barotropic and baroclinic parts:

\[U_0 = U_b + \tilde{U}_s \]

(56)

\[V_0 = V_b + \tilde{V}_s \]

\[W_0 = W_b + \tilde{W}_s \]

Where \(\int_{-1/\delta}^{0} (U_s, V_s) \, dz = 0 \)

(57)

We can use the Munk model and Ekman layer solution to calculate barotropic velocity components. By definition, we have:

\[\int_{-1/\delta}^{0} U \, dz = M_\lambda = \frac{\partial \psi}{\partial \phi} \]

(58)

\[M_\phi = \frac{1}{\cos \phi} \frac{\partial \psi}{\partial \lambda} = \int_{-1/\delta}^{0} V_0 \, dz + V_E \]

(59)

\[\psi = -\frac{1}{\cos \phi} \frac{\partial \psi}{\partial \lambda} \cos \phi f(\lambda) \text{ and } V_0 = -\frac{\tau_\lambda}{\sin \phi} \]

(60)

If we neglect the correction part due to the side layer in the Munk model then,

\[\int_{-1/\delta}^{0} V_b \, dz = V_m - V_E = -\frac{1}{\cos \phi} \frac{\partial \tau_\lambda}{\partial \phi} \cos \phi + \frac{\tau_\lambda}{\sin \phi} \]

(61)

Since \(V_b\) is uniform over the depth of \(Z\) so:

\[V_b = \delta \left( -\frac{1}{\cos \phi} \frac{\partial \tau_\lambda}{\partial \phi} \cos \phi + \frac{\tau_\lambda}{\sin \phi} \right) \]

(62)

Similarly:

\[\int_{-1/\delta}^{0} U \, dz = M_\lambda = \frac{\partial \psi}{\partial \phi} = \partial \left( -\frac{1}{\cos \phi} \frac{\partial \tau_\lambda}{\partial \phi} \cos \phi \right) \]

(63)

\[U_b = \delta \left( -\frac{1}{\cos \phi} \frac{\partial \tau_\lambda}{\partial \phi} \cos \phi \right) \]

(64)

The associated velocity \(W_b\) varies linearly with depth, from \(W_E\) at \(z=0\) to zero at the bottom then for \(W_b = W_E(1+\delta z)\)Then:

\[W_b = \frac{1}{\cos \phi} \frac{\sigma^* \partial (\tau_\lambda \cos \phi)}{\partial \phi} (1+\delta z) \]

(65)

The baroclinic velocity is the remaining part of the velocity field below the Ekman layer. The baroclinic current \(\tilde{U}_s, \tilde{V}_s\) has zero transport, and the vertical velocity \(\tilde{W}_s\) vanishes at both \(z=0\) and \(z = -1/\delta\). Since, we are interested in the velocity in the interior flow then, we start with the geostrophic balance:

\[-v_E \sin \phi = \tau_\lambda \]

(66)

\[-V_\sin \phi = \frac{-1}{\cos \phi} \frac{\partial \psi}{\partial \lambda} \]

(67)

\[\frac{\partial \psi}{\partial z} = T_0 \]

(68)

Taking a \(Z\) derivative of equations (65) and (66) and using equations (67) and (46) gives:

\[\frac{\partial U}{\partial z} = \left. \frac{\partial \psi}{\partial \phi} \right|_{\lambda} \frac{\sin \phi/2}{\sin \phi} - \cos \phi \frac{\sin \phi/2(Z + 1/\delta)}{\sin \phi/2(1/\delta)} - T_0 + \frac{1}{\sin \phi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a(n, m) \sin((n-1/2)\pi \cos(m-1)\pi \delta z \sin(n-1/2)\pi \phi) \]

(69)

And

\[\frac{\partial V}{\partial z} = \frac{\partial \psi}{\partial \lambda} \frac{\sin \phi}{\sin \phi} \cos \phi f(\lambda) \]

(70)

then

\[\tilde{U}_s = \frac{2}{\pi \sin \phi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a(n, m) \sin((n-1/2)\pi \cos(m-1)\pi \delta z \sin(n-1/2)\pi \phi) \]

(71)

Similarly:

\[\tilde{V}_s = 0 \]

(72)

Using the equation of incompressibility gives:

\[\frac{\partial \tilde{W}_s}{\partial z} = 0 \]

(73)

Now, since \(\tilde{W}_s\) vanishes at the top and the bottom, then:

\[\tilde{W}_s = 0 \]

(74)
The wind stress $\tau^\lambda$, the meridional Ekman transport $-\frac{\tau^\lambda}{\rho}$ and the derivatives of these (representing the wind stress curl and the Ekman vertical velocity) are shown in Figure 3. The stress and heat flux are defined as kinematical values (stress divided by density, heat flux divided by heat capacity per unit volume), making their dimensions equal to a velocity square and temperature-time velocity, respectively. The non-dimensional stream function transport (Fig. 4) is defined by equations of (58) to (60).

As shown in Figure 5, with increasing depth the speed reduces in about the 15 degree maximum amount (about 1/1) and the maximum speed at the equator is the 996/0.

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Fig. 3: (a) Variation with $\phi$ (latitude) of the wind stress $\tau^\lambda$, (b) the Ekman transport $-\frac{\tau^\lambda}{\rho}$, (c) the wind stress curl $\tau^\phi$, (d) the Ekman vertical velocity($-\frac{\tau^\lambda}{\rho}$). The values are non-dimensional.

Fig. 4: The normalized total transport stream function [Ekman transport plus baroclinic transport] in a horizontal plane. In the tropical gyre the transport between two neighboring streamlines is 0.3.

Fig. 5: The baroclinic surface current $U(\phi,Z)$. 
3.3. The first-order temperature: Transformation to an interior problem

The first order correction $T_1$ to the temperature field satisfies below equation.

$$\nabla^2 T_1 = \frac{u_0}{\cos \phi} \frac{\partial T_0}{\partial \lambda} + V_0 \frac{\partial T_0}{\partial \phi} + w_0 \frac{\partial T_0}{\partial z}$$  

(75)

Since $T_0$ is independent of $\lambda$ this equation takes on the simpler form:

$$\frac{\partial^2 T_1}{\partial z^2} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial T_1}{\partial \phi} \right) + \frac{1}{\cos^2 \phi} \frac{\partial^2 T_1}{\partial \lambda^2} =$$  

$$V_0 \frac{\partial T_0}{\partial \phi} + W_0 \frac{\partial T_0}{\partial z}$$  

(76)

With the right-hand side, a function of $z$ and $\phi$ only. The boundary conditions are homogeneous, i.e.,

$$\frac{\partial T_1}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = -\frac{1}{\delta}$$  

$$\frac{\partial T_1}{\partial \lambda} = 0 \quad \text{at} \quad \lambda = 0 \quad \text{and} \quad \lambda = 1$$  

(77)

$$\frac{\partial T_1}{\partial \phi} = 0 \quad \text{at} \quad \phi = 0,1$$

For better understanding, we suppose the total problem as the sum of four sub-problems:

$$T_1 = T_{11} + T_{12} + T_{13} + T_{14}$$  

(78)

1. $\nabla^2 T_{11} = V_0 \frac{\partial T_0}{\partial \phi} + w_0 \frac{\partial T_0}{\partial z}$  

(79)

And all boundary conditions are homogeneous. This solution reflects the effect of the interior solution.

2. $\nabla^2 T_{12} = 0$  

(80)

$$\frac{\partial T_{12}}{\partial z} = \sigma^* \frac{\partial}{\partial \phi} \frac{\partial T_0}{\sin \phi} \quad \text{at} \quad z = 0$$  

(81)

With the boundary condition (81) and other homogeneous conditions, $T_{12}$ indicates the effect of the Ekman layer advection.

3. $\nabla^2 T_{13} = 0$  

(82)

With the boundary condition:

$$\frac{\partial T_{13}}{\partial \lambda} = \sigma^* \delta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B(n,m) \cos \left( n - \frac{1}{2} \right)$$  

(83)

$$\pi \phi \cos (m-1) \pi \delta z \quad \text{at} \quad \lambda = 0$$

And other conditions are homogeneous. $T_{13}$ reflects the effect of western boundary current advection.

4. $\nabla^2 T_{14} = 0$  

(84)

With the condition:

$$\frac{\partial T_{14}}{\partial \lambda} = \sum \sum D(n,m) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos s(m-1) \pi \delta z$$  

(85)

at $\lambda = 0$ and $\lambda = 1$

and other conditions are homogeneous. $T_{14}$ indicates the effect of baroclinic upwelling and downwelling near $\lambda = 0$ and 1.

3.3.1. The Interior Advection Solution

The interior advection solution $T_{11}$ has forcing which is independent of $\lambda$ and the governing equation can be written as:

$$\frac{\partial^2 T_{11}}{\partial z^2} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial T_{11}}{\partial \phi} \right) = V_0 \frac{\partial T_0}{\partial \phi} + W_0 \frac{\partial T_0}{\partial z}$$  

(86)

Where $T_0, V_0$ and $W_0$ are defined as equation (47) and (87) to (88), respectively.

$$V_0 = \sigma^* \left( \frac{1}{\cos \phi} \left( \sin k \phi + \kappa \phi \cos k \phi - \frac{\phi \sin k \phi}{\cos \phi} \right) \right) \delta$$  

(87)

$$W_0 = \sigma^* \left( \frac{1}{\sin \phi} \left( \sin k \phi + \kappa \phi \cos k \phi - \frac{\phi \sin k \phi}{\sin \phi} \right) \right) (1 + \delta) \delta$$  

(88)

The solution to equation (49) is:

$$T_{11} = \sigma^* \delta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_1(n,m) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos [(m-1) \pi \delta Z]$$  

(89)

By substituting equation (89) into equation (86), Fourier decomposing in the $\phi$ and $Z$ directions and solving the resulting equations, $C_1(n,m)$ are obtained. According to equation (86), the nonlinear interior temperature advection to first order is due to the meridional and vertical (barotropic) motions. Figure 6a indicates the meridional temperature advection and figure 6b shows the vertical temperature advection in the interior, i.e. the right hand side of equation (86).

Considering equation (86) for $T_{11}$, one can see a positive advection term. Since $T_{11}$ is negative, the net effect is a cooling of the water.
3.3.2. The Ekman Transport

Next, consider the Ekman transport induced temperature \( T_{12} \). The non-homogeneous boundary condition at \( Z=0 \) is independent of \( \lambda \); the governing equation is as follows:

\[
\frac{\partial^2 T_{12}}{\partial Z^2} + \frac{1}{\cos \phi \frac{\partial}{\partial \phi} (\cos \phi \frac{\partial T_{12}}{\partial \phi})} = 0 \tag{90}
\]

With the homogeneous and below boundary conditions:

\[
\frac{\partial T_{12}}{\partial Z} = \sigma^* \sum A_n \cos(n - \frac{1}{2})\pi\phi \quad \text{at } Z = 0 \tag{91}
\]

The solution to equation (50) is:

\[
T_{12} = \sigma^* \left( \frac{\delta Z^2}{2} + z \right) \sum A_n \cos \left( n - \frac{1}{2} \right)\pi\phi
+ \sigma^* \sum \sum c_{12}(n, m) \cos \left( n - \frac{1}{2} \right)\pi\phi \cos(m - 1)\pi\delta z \tag{92}
\]

The coefficients \( c_{12}(n, m) \) are obtained by substituting equation (92) into (90), by invoking the
orthogonally property of \( \cos \left( n - \frac{1}{2} \right) \pi \phi \) and \( \cos (m - 1) \pi \delta z \) and by solving the resulting equations for \( c_{12}(n, m) \).

The Ekman layer will transport warm water. Since \( T_0 \) decreases as \( \phi \) increases in the tropical gyre \( \frac{\partial T_{12}}{\partial \phi} < 0 \) and there is an upward advection of warm water as well as a diffusion of heat down from the Ekman layer. So, as is shown in Figure 7, Forcing function at the top Ekman layer is positive. As seen from Figure 8, the Ekman layer is a source of heat \( (T_{12} > 0) \).

3.3.3. Boundary current advection

Now, consider the western boundary current advection solution, \( T_{13} \). The governing equation defining \( T_{13} \) is as follows:

\[
\frac{\partial^2 T_{13}}{\partial z^2} + \frac{1}{\cos \phi \partial \phi} \left( \cos \phi \frac{\partial T_{13}}{\partial \phi} \right) + \frac{1}{\cos^2 \phi} \frac{\partial^2 T_{13}}{\partial \lambda^2} = 0 \tag{93}
\]

With the homogeneous and below boundary conditions:

\[
\frac{\partial T_{13}}{\partial \lambda} = \delta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma \ast B(n, m) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos (m - 1) \pi \delta z \text{ at } \lambda = 0
\]

The solution to equation (93) is:

\[
T_{13} = \sigma \ast \delta \left( \lambda - \frac{\lambda^2}{2} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B(n, m) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos (m - 1) \pi \delta z + \sigma \ast \delta \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{13}(n, m, k) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos (m - 1) \pi \delta z \cos (k - 1) \pi \tag{95}
\]

Where \( C_{13}(n, m, k) \) and \( B(n, m) \) are obtained by substituting equation (95) into equation (93), Fourier decomposing in the \( \phi \) and \( Z \) directions and solving the resulting equations for them.

We have \( \frac{\partial T_{13}}{\partial \lambda} < 0 \) at the edge of boundary current. The heat advected to higher latitudes by the boundary current diffuses into the interior. Since, \( \frac{\partial T_{13}}{\partial \lambda} \) is small and the transport is also small, \( T_{13} \) is positive everywhere. So, the net effect is warming of the water everywhere, as seen in Figure 9a, b and c.
Fig. 9: $T_{12}$ field in a meridional section. (a) $\lambda=0$, (b) $\lambda=0.5$, (c) $\lambda=1$
3.3.4. Coastal up and downwelling

The coastal upwelling induces the solution \( T_{14} \). The equation governing \( T_{14} \) is

\[
\frac{\partial^2 T_{14}}{\partial Z^2} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial T_{14}}{\partial \phi} \right) + \frac{1}{\cos^2 \phi} \frac{\partial^2 T_{14}}{\partial \lambda^2} = 0 \tag{96}
\]

With the homogenous and below boundary conditions:

\[
\frac{\partial T_{14}}{\partial \lambda} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D(n, m) \cos \left( n - \frac{1}{2} \right) \pi \cos \phi \cos \left( m - \frac{1}{2} \right) \pi \delta z \cos \left( k - 1 \right) \pi \lambda \tag{97}
\]

The solution to equation (96) is:

\[
T_{14} = \lambda \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D(n, m) \cos \left( n - \frac{1}{2} \right) \pi \cos \phi \cos \left( m - \frac{1}{2} \right) \pi \delta z + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{14}(n, m, k) \cos \left( n - \frac{1}{2} \right) \pi \phi \cos \left( m - \frac{1}{2} \right) \pi \delta z \cos \left( k - 1 \right) \pi \lambda \tag{98}
\]

Where the values of \( D(n, m) \) and \( C_{14}(n, m, k) \) are obtained by substituting equation (98) into equation (96), using the orthogonally property of \( \cos \left( n - \frac{1}{2} \right) \pi \phi \) and \( \cos \left( m - \frac{1}{2} \right) \pi \delta z \) and solving the resulting equation for \( C_{14}(n, m, k) \).

The baroclinic upwelling and downwelling next to the meridional boundaries produce a positive value of \( \frac{\partial T}{\partial x} \) at the western and eastern sides (Figure. 10), i.e., cooling effects at the western boundary and heating at the eastern boundary. We find a diffusion heat from east to west.

4. Results and Discussion

To start the discussion of our results, let us determine the nondimensional coefficients \( E, R, \delta \) and \( \sigma^* \) and define a reasonable range for each, respectively. Let us choose the typical ocean depth, \( H \), to be 4000 m and the relative thermocline depth, \( \delta \), to be \( \pi \delta /30 \approx 0.105 \). Since, \( \delta = \frac{D}{R} \), we will have \( D \approx 420 \), which seems acceptable. From relation, we have \( Q_0 = (A_v \Delta T / D) \) where \( Q_0 \) the amplitude of the kinematic heat flux stands. If we let \( Q_0 = 1.7 \times 10^{-2} \text{cm}^{-1} \) corresponding to a heat flux of 150 cal \( \text{cm}^{-2} \text{day}^{-1} \), we find \( A_v \approx 3.6 \text{cm}^2 \text{sec}^{-1} \). From relation (27), we have \( \delta^2 = (A_v / A_H) (R/H)^2 \) where \( R \), the radius of the earth, is 6400 km. we then have \( A_H \approx 8.5 \times 10^8 \text{cm}^2 \text{sec}^{-1} \) which is large but reasonable.

We have \( E_H = (A_H / f_0 R^2) \) with \( f_0 = 10^{-4} \text{sec}^{-1} \) we find that \( E_H \approx 2 \times 10^{-5} \) and \( U_t = (g \alpha \Delta T / f_0 R) \approx 1.3 \text{cm} \text{sec}^{-1} \) where \( g \alpha = 0.1 \text{cm} \text{sec}^{-2} \text{K}^{-1} \). From the definition of the Rossby Number, we have \( R_0 = (U_t / f_0 R) \approx 2.05 \times 10^{-5} \). We choose \( \tau_0 = 2 \text{cm}^2 \text{sec}^{-2} \).

4.1. Velocity

Because of the wind stress, a complete mass transport gyre is generated. We reworked the Munk model in spherical system and then showed the streamline in (Figure. 3) for case \( E = 2 \times 10^{-5} \). This gyre results from the stress field. In the tropical gyre, the transport between two neighboring streamlines is 9.633 (10 12) cm3/sec.

The surface velocity resulting from the spherical case increases northward. At low latitudes, the velocity in the spherical system velocity has a minimum of 0.9 and maximum of 1.1 (Figure. 5).

4.2. Temperature

From the zeroth order solution, we display the temperature field which is produced by diffusive balance. Heating over the ocean surface diffuses throughout ocean. The zeroth order temperature field has a smooth meridional variation, with the isotherms starting out horizontally from the equator, as shown in Fig. 1.
The first order temperature field has been divided into four sub-problems as described in the former section. The nonlinear interior temperature advection to first order is due to the barotropic velocity. Figure 7a indicates the meridional temperature advection forcing function and Figure 7b shows the vertical temperature advection forcing function in the interior (the right hand side of equation (75). Those two effects counteract each other. Since, the vertical advection is stronger than meridional advection, the major effect will be due to vertical advection. Figure 7c gives the total interior advective forcing. Interior vertical advection has a strong effect. Considering equation (75) for T11, one can see that a positive advection term corresponds to a
heat sink in the first order calculation and because of a strong, unbalanced interior heat sink at low latitudes. T11 is negative so the net effect is a cooling of water everywhere. The Ekman layer will transport warm water from equator to higher latitudes. As is shown by Figure 6, the corrected boundary condition is positive at low latitudes. As seen from Figure 8, at low latitudes, the Ekman layer is a source of Heat (T2>0).

The western boundary current advection is barotropic. Since $\frac{\partial T_3}{\partial x}$ is small, the transport is also less. T13 is positive everywhere so the net effect is a warming of the water everywhere, as seen in Figures 9a, 9b and 9c. The baroclinic upwelling and downwelling next to the meridional boundaries produces a positive value of $\frac{\partial T}{\partial x}$ at the western and eastern sides. i.e., cooling effects are evident at the western boundary and heating effects are evident at the eastern. This results in a diffusion of heat from east to west. Figure 10 shows negative isotherms which indicate cooling. At $\lambda=0$, for $\lambda=1$, it is identical to Figure 10c, the values of Isotherms are positive, which indicates warming there.

4.3. Wind Induced Temperature

If we add the three wind induced temperature corrections together (T11+T12+T13), at low latitudes there is a cooling due to interior advection (see Fig.11).

![Fig.11: The solution to problems T1+T2+T3 for (a) $\lambda=0$, (b) $\lambda=0.5$ and (c) $\lambda=1$](image-url)
4.4. Wind Induced and Up and Downwelling Effect

If we add the baroclinic upwelling and downwelling to the first three results we will have an additional cooling along the western side. This is seen in Figure 12.

4.5. Total Field

The total field, $T_0 + T_1$ for $\lambda=0$, shown in Figure 13, is reasonable. At $\lambda=0$ the isotherms are shallow at the equator.

Fig. 12: the solution to problems $T_1 + T_2 + T_3 + T_4$ for (a) $\lambda=0$, (b) $\lambda=0.5$ and (c) $\lambda=1$
Figure 14. shows the surface temperature field, T₀+T₁R₀E⁻¹. In the tropical region, the eastern surface water is warmer than the western surface water, since in the tropical region there is a downward slope of the isotherms to the north, we will have a westward baroclinic current in the upper layer at low-latitudes.

![Fig. 14: Total solution T₀+T₁R₀E⁻¹ at surface.](image)

**Conclusions**

We formulated a mathematical model and studied a problem involving the calculation of the three dimensional temperature and velocity fields forced by a meridionally varying wind stress and surface heat flux. We derived by this model, the first interior temperature field into four subproblems as follows: the effect of geostrophic convection, the effect of top Ekman layer, the effect of the barotropic western boundary current advection, and the effect of coastal baroclinic upwelling and downwelling. In this model, we computed velocity field by first computing the vertically averaged flow, then the top Ekman layer flow and finally, the difference between the two, which is the geostrophic flow.

**References**


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